Wake Up and Smell the Coffee

Rosalie A. Dance and James T. Sandefur

References

Dance, R. and J. Sandefur, Reading this could help you sleep, www.georgetown.edu/projects/handsonmath

Dance, R. and J. Sandefur, So much coffee, so little time, www.georgetown.edu/projects/handsonmath


Around the world, people enjoy caffeine in both food and drink. Caffeine is an alkaloid compound that comes from plants, including coffee, tea, kola nuts, maté, cacao, and guarana. Some people drink caffeine drinks because they like the taste, and some drink them for the physical effect they can have. Most people are aware of differences in the way they feel as a result of drinking caffeine; it stimulates the central nervous system, the heart, and the respiratory system. For many people, the effect is pleasant and energizing, a “wake up” or a “pick-me-up,” and a way to delay fatigue. But for others, the effects are unpleasant. Laboratory tests indicate that one to three cups of coffee can increase a person’s capacity for sustained intellectual effort and decrease reaction time (that’s good) but may adversely affect the ability to perform tasks involving delicate muscular coordination and accurate timing (that’s bad).

In this article, we explore the dynamics of caffeine in the body through the use of exponential functions.

The amount of caffeine in food or drink can vary. Manufactured products such as sodas and candy bars have consistent levels of caffeine, but coffee and tea and other drinks have varying amounts. The “approximate average” amounts given in the tables at the left give us a reasonable picture, however.

The effects of caffeine can only be felt when sufficient amounts are present. For most people, 32 to 200 mg of caffeine acts as a minor stimulant; these amounts have been shown to speed up reactions in simple routinized tasks in a laboratory setting. As noted above, such minor stimulant effects are experienced by some people as pleasant, but for others they are unpleasant. Steadiness of the hand has been shown to deteriorate when there is more than 200 mg of caffeine present. More than 300 mg is enough to produce temporary insomnia. 480 mg has been shown to cause panic attacks in panic disorder patients. Amounts of 5 to 10 grams (5,000–10,000 mg) of caffeine cause death.

Chemicals are eliminated from the body in two primary ways: (1) through filtration by the kidneys, and (2) metabolism by enzymes from the liver. The kidneys tend to filter out a constant proportion of the chemical, that proportion depending on what the chemical is and the individual. In the “average person,” about 13% of the caffeine present at the beginning of each hour is eliminated during the hour. So:

### Table: Amount of Caffeine in a 12-Oz Soda

<table>
<thead>
<tr>
<th>Soda Type</th>
<th>Caffeine Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coca Cola</td>
<td>45.6 mg</td>
</tr>
<tr>
<td>Diet Coke</td>
<td>45.6 mg</td>
</tr>
<tr>
<td>Surge</td>
<td>51 mg</td>
</tr>
<tr>
<td>Dr. Pepper</td>
<td>39.6 mg</td>
</tr>
<tr>
<td>Pepsi</td>
<td>37.2 mg</td>
</tr>
<tr>
<td>Diet Pepsi</td>
<td>35.4 mg</td>
</tr>
<tr>
<td>Mountain Dew</td>
<td>55 mg</td>
</tr>
</tbody>
</table>

### Table: Amount of Caffeine in a 1.5-Oz Chocolate Bar

<table>
<thead>
<tr>
<th>Chocolate Type</th>
<th>Caffeine Amount</th>
</tr>
</thead>
</table>
| Hershey’s Special, dark chocolate | 31 mg
| Hershey Bar, milk chocolate    | 10 mg

### Table: Amount of Caffeine Present in Body and Effect

<table>
<thead>
<tr>
<th>Caffeine Amount</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>32–200 mg</td>
<td>Minor stimulant</td>
</tr>
<tr>
<td>&gt;200 mg</td>
<td>Unsteady hands</td>
</tr>
<tr>
<td>&gt;300 mg</td>
<td>Temporary insomnia</td>
</tr>
<tr>
<td>480 mg</td>
<td>Panic attacks in panic disorder patients</td>
</tr>
<tr>
<td>5000 mg</td>
<td>May cause death</td>
</tr>
</tbody>
</table>

You Try It #1

If a person starts the day by drinking 3 cups of coffee (one right after the other) containing 130 mg of caffeine each, assuming the caffeine is absorbed into the body immediately,

a. How much caffeine will there be in this person’s body 1 hour later? 2 hours later? 3 hours later?

b. Reflect on how you computed the answers to part (a) to help you develop a simple expression that gives the amount of caffeine in this person’s body 24 hours later, assuming no additional caffeine is consumed. Find the amount. You do not need to compute the amount of caffeine in the body one hour after another until you reach 24; you can find this amount directly.

Let’s explore this further.
You Try It #2

If a person starts the day by drinking 3 cups of coffee containing 130 mg of caffeine each, assuming the caffeine is immediately absorbed into the body,

a. Write an equation in the form \( y = f(t) \) for the amount of caffeine in the body \( t \) hours after ingesting 390 mg.

b. Graph the function you developed in part (a).

c. What is the long term behavior ("end behavior") of the function you graphed? What does that say about the 390 mg of caffeine the person drank in the morning?

d. How much caffeine will be in the person’s body 30 minutes after drinking 3 cups of coffee (when \( t = ? \))?

e. How long will it take for the 390 mg of caffeine in the person’s body to be reduced to 195 mg?

Read the definition of half-life in the margin carefully. The concept of half-life is applied to elimination of drugs from the body, and to radioactive decay, and to other situations. Note that half-life is a measure of time. It takes about 5 hours for 390 mg of caffeine to be reduced by half to 195 mg. Thus the half-life of caffeine in the body is about 5 hours.

You Try It #3

To understand half-life, first determine how much of the original 390 mg of caffeine is in the body after two half-lives; then after three half-lives. Then determine the value of \( t \), time, for each of these answers. Answer the questions in the order they are asked; it will help you get a deeper understanding of the half-life concept.

Now we get personal.

You Try It #4

a. Make a table like the one at left to keep a record of your own caffeine consumption over a 24-hour period beginning when you wake in the morning. If you do not eat or drink caffeine, use data from a friend. Estimate the amount of caffeine in the drinks or chocolate you consume using the information in the margin of page 1 or from product labeling. Assume that the caffeine is consumed and absorbed by your body immediately, even though you may actually linger over your drink and even though it will actually take a short time for your body to absorb the caffeine.

b. From your table, sketch a graph of the amount of caffeine in your body (or your friend’s) over this 24-hour period.

c. Use your graph to help you write a conditional function (a piecewise-defined function) that shows the amount of caffeine in your body at any time, \( t \), during the day. To write a function using a single variable for time, note that \( t \) measures the number of hours after some initial starting time.

d. Throughout the day observe how you feel. Do you notice any patterns relating your caffeine level and how you feel? Comment on what you observe.
Filtration by the kidneys is one of the major processes by which our bodies eliminate chemicals. The kidneys usually filter a fixed fraction of the chemical present during each hour (or each day, or each year). Caffeine, lead, and cadmium are all eliminated this way.

For caffeine, we measure \( t \) in hours because caffeine is eliminated from the body fairly quickly. For lead, we would measure \( t \) in days. Cadmium (a chemical in cigarettes) is eliminated so slowly that we could measure \( t \) in years when we model cadmium elimination.

Another process of elimination is metabolism by enzymes from the liver. This method often results in a nearly constant amount of the chemical being broken down in a given time period, instead of a fixed fraction of it as above. Alcohol is eliminated from our bodies this way. We cannot use an exponential function to model elimination of bodies this way. We cannot use an exponential function to model elimination of alcohol; a rational function works well for that model. See Consortium 66 for a study of such a model.

The discrete model might be written this way:

\[
\begin{align*}
C(0) &= 130 \\
C(n) &= 0.87C(n-1) + 130
\end{align*}
\]

Is this about what you did with your calculator? What was \( C(n-1) \) in your calculator computation?

If you didn’t use 0.87, could you have shortened your calculator formula by using it instead of what you did?

We used 0.87\(C(n)\) because it is shorter than \(C(n) - 0.13C(n)\).

These computations can also be done using a spreadsheet in which the formula in cell A2 below is copied down as far as necessary.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>130</td>
</tr>
<tr>
<td>2</td>
<td>0.87*A1+130</td>
</tr>
<tr>
<td>3</td>
<td>0.87*A2+130</td>
</tr>
<tr>
<td>4</td>
<td>0.87*A3+130</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You have seen by now that a process that eliminates a fixed fraction of a chemical from our bodies each hour can be modeled by an exponential function of the form \( f(t) = Ar^t \). Here \( t \) is a length of time since the start, \( A \) is the amount of the chemical present at the start, and \( r \) represents the fraction of the chemical that remains in our body (the fraction that is not eliminated during the hour).

You probably used a piecewise-defined function with “pieces” involving the form \( Ar^t \) to model your own caffeine intake and retention. Suppose someone drank one cup of coffee every hour all day long. This might be unrealistic where coffee drinking is concerned, but the approach we develop in this model applies to medicines that are taken every 4 hours, for example, and to cadmium in people who smoke every day, or toxic chemicals in an industrial environment, and much more.

So imagine, please, a person involved in a high-risk situation who needs to stay awake and working for 48 hours who compulsively gulps down one cup of coffee every hour on the hour beginning at 6 a.m. As before, we assume that each cup contains 130 mg of caffeine, and that 13% of the coffee present in his body at the start of each hour is eliminated during the hour.

You Try It #5

Explore the dynamics of the caffeine using a graphing calculator.

a. To begin, input the first 130 mg for the 6 a.m. coffee on the homescreen. Now calculate the amount of coffee still present after the second (7 a.m.) cup of coffee. And after the third cup. And after the fourth cup.

b. Did you find yourself repeating the same process over and over? Write down what you did at each stage. A graphing calculator (or computer spreadsheet program) can do this repetitive work for you. Start again. Input 130 as before, and ENTER. Use the Answer key (ANS) to represent the amount that was present just after drinking the last cup of coffee, and input a formula that will repeat the process of calculation that you did at each stage. Then just hit ENTER repeatedly to find out how much caffeine was present after 2, 3, and 4 cups of coffee (that is, at 7 a.m., 8 a.m., and 9 a.m.).

If you wanted, you could use this process to find out how much caffeine is there at the end of every hour right through this sleepless knight’s 48th hour. This process of iteration of a function, using your most recent output as input at the next stage, gives us a discrete model of the process that is occurring in the person’s body. It allows us to observe the process in steps.

But let’s try to write a “closed form” function for the every-hour-on-the-hour coffee drinker, a function more like the one we used for the caffeine drinkers in the first part of our article, \( f(t) = Ar^t \).

The big difference between this situation and the one we modeled before is the addition of a new 130 mg of coffee each hour. Perhaps, then, the new situation could be modeled by \( f(t) = Ar^t + C \).

Because 13% of the caffeine is eliminated each hour, and thus 87% is retained, we know that the value of \( r \) in this model should be 0.87, just as before. But, perhaps surprisingly, in this new situation, \( A \) is not the initial amount of caffeine, from the first cup, and \( C \) is not the amount added with each new cup, either. The form of the function is right, but we have to calculate \( A \) and \( C \) from the basic premises of the situation.
Taking 6 a.m. as 0 hour (that is, \( t = 0 \) represents 6 a.m. on the first day), we know that \( f(0) = 130 \) because that is when he drank his first cup of coffee. Substituting this into our proposed model, we have \( f(0) = 130 = A r^0 + C \), or \( A + C = 130 \).

When \( t = 1 \), \( f(1) = (0.87)(130) + 130 = 243.1 \). Again by substitution into our model, we see that \( 243.1 = A (0.87)^1 + C \). That is, \( 0.87A + C = 243.1 \).

With these two results, we can determine the values of \( A \) and \( C \) for our function, \( f(t) = A(0.87)^t + C \).

**You Try It #6**

a. Solve the pair of equations \( A + C = 130 \) and \( 0.87A + C = 243.1 \) for \( A \) and \( C \). Use your results to write the function \( f(t) = A r^t + C \) that models the amount of caffeine in the body of the person who drank one cup of coffee each hour, each cup containing 130 mg of caffeine, with the body eliminating 13% of the caffeine present each hour.

b. Evaluate your function at \( t = 2 \) and at \( t = 3 \). \( f(2) \) should give you the amount of caffeine in the body 2 hours after he started drinking coffee, thus, just after he drank his third cup. Does your value for \( f(2) \) agree with the amount of caffeine you calculated should be present after the third cup in YTI#5?

c. Evaluate \( f(3) \). Compare the result to the corresponding result in YTI#5 and explain what this result tells us.

A good model illustrates long term behavior as well as giving the status at particular input values.

**You Try It #7**

a. Graph the function you developed in YTI#6. Find a window that gives you a good view of its long term behavior.

b. After studying the graph, describe what eventually happens to the amount of caffeine in a person’s body if he continues drinking 1 cup of coffee each hour, not just for two days, but indefinitely, day after day after day.

c. 5000 mg of caffeine in a person’s body can cause death. That’s scary. Will a person who drinks a cup of coffee every hour, with 130 mg in each cup, eventually have 5000 mg of caffeine in his/her body?
1 a  390 \times 0.87 = 339.3 \text{ mg 1 hour later}  \\
390 \times 0.87^2 = 295.2 \text{ mg 2 hours later}  \\
390 \times 0.87^3 = 256.8 \text{ mg 3 hours later}  \\
390 \times 0.87^{24} = 13.8 \text{ mg 24 hours later}  \\

b  
Discussion will probably show that students computed these results in various ways. Be sure that all students see (i) the relationship between eliminating 13% and retaining 87% and (ii) that multiplying again by 0.87 is equivalent to increasing the exponent on 0.87 by 1.

2 a  
f(t) = 390(0.87)^t  

b  
As \( t \) increases, the function values approach 0.  
This illustrates that if no more caffeine is taken into the body, the amount of caffeine remaining dissipates to none.

d  
f \left( \frac{1}{2} \right) = 390 \sqrt[2]{0.87} \approx 364 \text{ mg. Students can read this from their graph or calculate it.}

e  
About 5 hours. Students can read this from the graph.  
On the graph of \( f(t) = 390(0.87)^t \) shown for part (b), we also show the line \( y = 195 \). The intersection of the two graphs provides the answer to the question.

3  
After two half-lives, 390 mg has been reduced to  
\[
\left( \frac{1}{2} \right)^2 \times 390 = 97.5.
\]

After three half-lives, 390 mg has been reduced to  
\[
\left( \frac{1}{2} \right)^3 \times 390 = 48.75. \text{ Thus, } t \approx 10 \text{ is 2 half-lives, and } t \approx 15 \text{ is 3 half-lives.}
\]

Students can read these results from their graphs. Be sure they recognize that after each 5-hour period (approximately), the amount of caffeine remaining in the body is about half what it was at the beginning of the 5-hour period. Studying the graph with this in mind will help students understand this useful concept. The three points are labeled H1, H2, and H3 on our graph.

4  
Each person’s table and graph will be different, of course, but we suppose that many will have more than one drink or chocolate bar containing caffeine during the day. These students will have to write and graph a conditional function.

As an example, a person who consumes 130 mg at 8 a.m., 40 mg at 9:30 a.m., and 165 mg at 3 p.m., but no more during the 24-hour period could write

\[
\begin{align*}
0 \leq t < 1.5 & \quad 130 \cdot (0.87)^t \\
1.5 \leq t < 7 & \quad 130 \cdot (0.87)^t + 40 \cdot (0.87)^{t-1.5} \\
7 \leq t < 24 & \quad 130 \cdot (0.87)^t + 40 \cdot (0.87)^{t-1.5} + 165 \cdot (0.87)^{t-7}
\end{align*}
\]

or

\[
\begin{align*}
0 \leq t < 1.5 & \quad 130 \cdot (0.87)^t \\
1.5 \leq t < 7 & \quad 145.49 \cdot (0.87)^{t-1.5} \\
7 \leq t < 24 & \quad 232.64 \cdot (0.87)^{t-7}
\end{align*}
\]

5 a  
After \( n \) cups, \( t \) hours after the first cup, there will be \( f(t) \) mg of caffeine in the body.

<table>
<thead>
<tr>
<th>( n ) cups</th>
<th>( t ) hours</th>
<th>( f(t) ) mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>130</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>243.1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>341.5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>427.1</td>
</tr>
</tbody>
</table>

b  
The calculator commands are as follows:

\[
\begin{align*}
\text{130 ENTER} \\
\text{0.87ANS + 130 ENTER} \\
\text{ENTER} \\
\text{ENTER}
\end{align*}
\]

Students should note that the calculator performs the iterations just as described by the notation for the discrete system, for example as given in the margin.

6 a  
Solving the system of equations gives \( A = -870, C = 1000. \) Thus, the function is \( f(t) = -870(0.87)^t + 1000. \)

b  
\[
f(2) = -870(0.87)^2 + 1000 \approx 341.5,
\]
which should agree with the result in YTI#5.

c  
\[
f(3) = 427.1,\text{ agreeing with the corresponding result in YTI#5.}
\]

7 a  
\[
f(t) = -870(0.87)^t + 1000.
\]

b  
The caffeine in the body approaches an upper boundary of 1000 mg.

c  
No. The caffeine level will not exceed 1000 mg.